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Hidden symmetries of the motion of a charged particle in an uniform magnetic field

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Abstract : It is shown that, by means of an appropriate canonical transformation, the constants of motion for a charged particle in a uniform magnetic field can be associated with ignorable variables in the Hamiltonian. The canonical coordinates where the hidden symmetries of the Hamiltonian become obvious are related with the generating functions of the rigid translations.

Keywords : Hidden symmetries, Hamiltonian formalism.

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1. Introduction

As is well known, the existence of ignorable coordinates in the Hamiltonian of a mechanical system allows one to find constants of motion in a straightforward manner. Each ignorable coordinate, as well as each constant of motion that does not depend explicitly on the time, is associated with a one-parameter group of canonical transformations that leaves the Hamiltonian invariant. However, in some cases, the existence of several one-parameter groups of symmetries of the Hamiltonian cannot be associated with the simultaneous existence of ignorable coordinates. A simple example of this situation is given by the problem of a particle in a central force field; the Hamiltonian is invariant under the rotations about all axes passing through the center of force, but the number of ignorable coordinates is 1 at most. Nevertheless, all components of the angular momentum are conserved as a consequence of the mentioned invariance.

In some other cases, there may be constants of motion that are not associated with transformations on the configuration space. For instance, one can readily verify that in the case of a particle in a uniform gravitational field, with Hamiltonian

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$$H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+mgy\,, \tag{1}$$

where g is the acceleration of gravity, p_x is a constant of motion (associated with the ignorable coordinate x) and $A\equiv p_x p_y+m^2 gx$ is also conserved. However, the conservation of A is not related with the invariance of H under transformations on the phase space induced by transformation of the configuration space. In fact, for any mechanical system, the constants of motion associated with the invariance of the Hamiltonian under transformations induced by groups of transformations on the configuration space are homogeneous functions of degree 1 of the canonical momenta.

The problem of a classical charged particle in a uniform magnetic field also involves constants of motion not related to transformations on the configuration space (see Section 2). Following the procedure employed in Ref. [1], we find that expressing the Hamiltonian in terms of an appropriate set of canonical coordinates, the existence of these constants of motion follows from the existence of a pair of ignorable canonically conjugated variables Apart from constant factors, the new canonical coordinates are the components of the kinematical momentum and the generators of translations, which would coincide with the former, if the magnetic field is absent.

2. Hidden and obvious symmetries

The vector potential for a uniform magnetic field $\boldsymbol{B}=B\hat{z}$ can be chosen as

$$A=\frac{1}{2}B\times r=\frac{1}{2}B\left(-y\hat{x}+x\hat{y}\right). \tag{2}$$

Therefore, the Hamiltonian for a charged particle of mass m and electric charge q in this uniform magnetic field can be taken as

$$H=\frac{1}{2m}\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right)^2=\frac{1}{2m}\left[\left(p_x+\frac{qB}{2c}y\right)^2+\left(p_y-\frac{qB}{2c}x\right)^2+p_z^2\right. \tag{3}$$

(see, *e.g.*, Refs. [2,3]), making use of the standard symplectic structure of the phase space, where p_x, p_y, and p_z are the canonically conjugate momenta of the Cartesian coordinates x, y, and z, respectively. In this manner, only the coordinate z is ignorable (and its conjugate momentum, p_z, is conserved).

According to Hamilton's equations, from eq. (3) it follows that $\boldsymbol{p}=m\dot{\boldsymbol{r}}+(q/c)\boldsymbol{A}$. Since the vector potential is not uniquely determined by a given magnetic field, the canonical momenta turn out to be gauge-dependent; this minor drawback is a consequence of employing the canonical formalism. (However, it is also possible to give a gauge-independent Hamiltonian formulation of the interaction with a magnetic field, see, *e.g.*, Ref. [4] and the references cited therein.) In what follows we will make use of the standard Hamiltonian formalism in order to show, in this usual context, how to exhibit the symmetries of the Hamiltonian (3), employing canonical coordinates. In Section 4 we will discuss the effect of the gauge transformations.

One can readily verify (making use of the Hamilton equations or of the Poisson brackets) that

$$K_1 \equiv p_x - \frac{qB}{2c} y, \quad K_2 \equiv p_y + \frac{qB}{2c} x \tag{4}$$

are constants of motion. Since K_1 and K_2 are not homogeneous functions of degree 1 of the canonical momenta, they are not related with transformations on the configuration space, such as translations or rotations; they correspond to *hidden symmetries* of the Hamiltonian (3). As pointed out above, being constants of motion that do not depend explicitly on the time, K_1 and K_2 generate one-parameter groups of canonical transformations that leave the Hamiltonian invariant. The infinitesimal generators of these one-parameter groups are the Hamiltonian vector fields X_{K_1} and X_{K_2}, where for any differentiable function f,

$$X_f \equiv \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right), \tag{5}$$

and q_i, p_i is an arbitrary set of canonical coordinates. (Then, the infinitesimal generator of the group of canonical transformations generated by p_z is $\partial/\partial z$.) Thus,

$$X_{K_1} = \frac{\partial}{\partial x} + \frac{qB}{2c} \frac{\partial}{\partial p_y} \qquad X_{K_2} = \frac{\partial}{\partial y} - \frac{qB}{2c} \frac{\partial}{\partial p_x} \tag{6}$$

The fact that these vector fields involve $\partial/\partial p_x$ and $\partial/\partial p_y$ means that the canonical transformations generated by K_1 and K_2 do not come from transformations on the configuration space.

A straightforward computation shows that X_{K_1} and X_{K_2} commute, which also follows from the fact that the Poisson bracket $\{K_1, K_2\}$ is a constant,

$$\{K_1, K_2\} \equiv \sum_i \left(\frac{\partial K_1}{\partial q_i} \frac{\partial K_2}{\partial p_i} - \frac{\partial K_1}{\partial p_i} \frac{\partial K_2}{\partial q_i} \right) = -X_{K_1} K_2 = -\frac{qB}{c}. \tag{7}$$

(For any pair of functions, f, g, the commutator of the Hamiltonian vector fields X_f and X_g is related with the Hamiltonian vector field corresponding to their Poisson bracket by $[X_f, X_g] = -X_{\{f,g\}}$.)

Since the velocity of the particle along the z-axis is constant, we can restrict ourselves to the motion on the xy-plane. The Hamiltonian now will be

$$h = \frac{1}{2m}\left[\left(p_x + \frac{qB}{2c}y\right)^2 + \left(p_y + \frac{qB}{2c}x\right)^2\right] \tag{8}$$

and since X_{K_1} and X_{K_2} do commute, according to the Frobenius theorem, they are tangent to a family of two-dimensional surfaces that foliate the phase space. By inspection, or making use of the method of characteristics, one finds that the functions that simultaneously satisfy the conditions $X_{K_1} f = 0$ and $X_{K_2} f = 0$ (which are equivalent to $\{K_1, f\} = 0$ and $\{K_2, f\} = 0$ are arbitrary functions of

$$F_1 \equiv x - \frac{2c}{qB}p_y \quad \text{and} \quad F_2 \equiv y + \frac{2c}{qB}p_x \tag{9}$$

and, therefore, the two-dimensional surfaces mentioned above are given by F_1 = const, F_2 = const.

The Hamiltonian h obeys $X_{K_\alpha} h = 0$ (as a consequence of the conservation of the K_α), which implies that it must be a function of F_1 and F_2. In fact, one readily finds that

$$h = \frac{1}{2m}\left(\frac{qB}{2c}\right)^2 \left(F_1^2 + F_2^2\right). \tag{10}$$

On the other hand, one finds that the Poisson bracket $\{F_1, F_2)$ is also a constant,

$$\{F_1, F_2\} = \frac{4c}{qB} \tag{11}$$

(*cf.* eq. (7)). Since $\{F_1, F_2)$ is different from zero, F_1 and F_2 cannot be the momenta of a system of canonical coordinates, but, absorbing the constant factor appearing on the right-hand side of eq. (11) into F_1 and F_2, we can identify a pair of canonically conjugated variables. For example, we can take $u = (1/2)\,F_1$ and $p_u = (qB/2c)\,F_2$, that is

$$u \equiv \frac{x}{2} - \frac{c}{qB}p_y, \qquad p_u \equiv p_x + \frac{qB}{2c}y \tag{12}$$

In terms of these canonically conjugated variables, the Hamiltonian h is given by

$$h = \frac{p_u^2}{2m} + \frac{m}{2}\left(\frac{qB}{2c}\right)^2 u^2, \tag{13}$$

which has the standard form of the Hamiltonian of a one-dimensional harmonic oscillator (As is well known, the problem of a charged particle is a uniform magnetic field in quantum mechanics can be related with the problem of a one-dimensional harmonic oscillator)

Since we are considering now a four-dimensional phase space, in addition to u and p_u we have to choose another pair of canonically conjugated variables, v and p_v, say, which will not appear in the Hamiltonian h and, therefore, are constants of motion According to the derivations above, a simple choice is essentially given by K_1 and K_2,

$$v \equiv \frac{x}{2} + \frac{c}{qB} p_y \qquad p_x - \frac{qB}{2c} y \tag{14}$$

(see eqs (4) and (7))

In summary, (u, v, p_u, p_v) is a system of canonical coordinates in terms of which the Hamiltonian h, given by eq (13), possesses two ignorable coordinates, by contrast with eq (8), then explaining the existence of the constants of motion K_1 and K_2

We end this section showing that the existence of the constants of motion K_1 and K_2, together with the fact that h itself is conserved (since it does not depend explicitly on the time), allows us to find easily the orbit of the particle and the hodograph in terms of the original variables Indeed, combining eqs (4) and (8) we obtain

$$E = \frac{1}{2m}\left(\frac{qB}{c}\right)^2 \left[\left(x - \frac{c}{qB}K_2\right)^2 + \left(y + \frac{c}{qB}K_1\right)^2\right]$$

$$= \frac{2}{m}\left[\left(p_x - \frac{1}{2}K_1\right)^2 + \left(p_y - \frac{1}{2}K_2\right)^2\right] \tag{15}$$

where E is the value of h Thus, in the xy plane and in the $p_x p_y$ plane, the orbit is a circle and the values of the constants of motion K_1 and K_2 determine the centers of both circles

3. Relationship with the generators of translations

The elementary problem considered above is also interesting because of the difference between the canonical momentum and the kinematical momentum, mv In the Lagrangian or the Hamiltonian formalism, the interaction with a magnetic field makes it necessary to introduce the vector potential and, in Cartesian components, the canonical momentum turns out to be given by

$$p_i = mv_i + \frac{q}{c} A_i, \tag{16}$$

where $\mathbf{v}$ is the velocity of the particle

As pointed out in Ref [5], under rigid translations in the configuration space, the Cartesian components of the velocity of the particle must be invariant; therefore, if $\mathcal{P}_i$ denotes the generating function of rigid translations along the Cartesian axis x_i, we must have

$$\left\{mv_i, \mathcal{P}_j\right\} = 0\,, \qquad \left\{x_i, \mathcal{P}_j\right\} = \delta_{ij} \tag{17}$$

However, from eq (16) it follows that $\left\{mv_i, p_j\right\} = \left\{p_i - (q/c)\,A_i, p_j\right\} = -(q/c)\partial A_i/\partial x_j$, which may be different from zero, thus showing that, when there is a nonvanishing magnetic field, the canonical momentum is *not* the generating function of rigid translations

In terms of the notation and conventions employed in Section 2, one find that the x- and y-component of the kinematical momentum are

$$mv_x = p_u, \quad mv_y = -\frac{qB}{c}u \tag{18}$$

and that the Poisson bracket relations (17) are satisfied taking

$$\mathcal{P}_1 = p_v, \quad \mathcal{P}_2 = \frac{qB}{c}v \tag{19}$$

Thus, the canonical coordinates (u, v, p_u, p_v) introduced at the end of the preceding section are, essentially, the Cartesian components of the kinematical momentum and the generating functions of translations along the x- and y-axis

It may be noticed that, while the kinematical or the canonical momentum is not conserved $\mathcal{P}_1$ and $\mathcal{P}_2$ are constants of motion (which follows from the fact that the magnetic field itslef is invariant under rigid translations [5] or from their relation with the ignorable coordinates v and p_v given by eqs (19)) A somewhat unexpected result, since the group of translations of the plane is Abelian, is that the Poisson bracket between the generating functions of translations given by eqs (19) is different from zero, $\left\{\mathcal{P}_1, \mathcal{P}_2\right\} = -qB/c$ Nevertheless, as pointed out in Section 2, the infinitesimal generators of the transformations generated by them, as well as the corresponding one-parameter groups of canonical transformations, do commute In the quantum version of this problem, the nonvanishing of the commutator of the operators corresponding to $\mathcal{P}_1$ and $\mathcal{P}_2$ leads to a ray-representation of the group of translations of the plane [6]

4. Gauge transformations

Under a gauge transformation, the vector potential $\boldsymbol{A}$ is replaced by

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\xi\,, \tag{20}$$

where ξ is an arbitrary (differentiable) function of the coordinates, so that the magnetic field, $\boldsymbol{B}$, is left invariant. As pointed out above, the canonical momentum is given by $\boldsymbol{p} = m\dot{\boldsymbol{r}} + (q/c)\boldsymbol{A}$; therefore, the gauge transformation (20) is accompanied by the transformation

$$\boldsymbol{p}' = \boldsymbol{p} + \frac{q}{c}\nabla\xi . \tag{21}$$

Then

$$p'_x dx + p'_y dy + p'_z dz = p_x dx + p_y dy + p_z dz + d\left(\frac{q}{c}\xi\right),$$

which implies that the transformation (21), together with

$$\boldsymbol{r}' = \mathbf{r} ,$$

is a canonical transformation. Hence, the Poisson brackets calculated in terms of x, y, z, p_x, p_y, p_z, coincide with those calculated using $x, y, z, p'_x, p'_y, p'_z$ as canonical variables. In other words, *the underlying symplectic structure of the phase space is gauge-independent.* This means that we only have to make the replacement of the canonical momenta p_i by p'_i, according to (21), and all the Poisson bracket relations given above remain valid, as well as all the relations that do not contain explicitly the canonical momenta, such as eqs. (10), (13), (18), and (19).

For instance, in terms of $\boldsymbol{p}'$, the constants of motion K_1 and K_2 are given by

$$K_1 = p'_x - \frac{q}{c}\frac{\partial\xi}{\partial x} - \frac{qB}{2c}y , \qquad K_2 = p'_y - \frac{q}{c}\frac{\partial\xi}{\partial y} - \frac{qB}{2c}x ,$$

and, as one can readily verify, the Poisson bracket $\{K_1, K_2\}$, calculated using the new canonical variables, is equal to $-\, qB / c$ (*cf.* Eq. (7)).

5. Final remarks

In the example considered in this paper, two constants of motion, K_1 and K_2, not related with obvious symmetries of the original Hamiltonian, have been identified. Since the Poisson bracket $\{K_1, K_2\}$ is a constant different from zero, the existence of K_1 and K_2 can be seen as a consequence of a pair of ignorable canonically conjugated variables. A set of canonical coordinates where the symmetries become obvious is formed, essentially, by the Cartesian components of the kinematical momentum and the generating functions of translations, which are independent only if the magnetic field does not vanish.

In the example of a particle in a central force field, mentioned in the Introduction, not all components of the angular momentum can be associated with ignorable coordinates because the Poisson bracket of any two of these components is neither zero nor a constant.

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